J. Fluid Mech. (1965), *vol.* 22, *part* 4, *pp.* 753–761
Printed in Great Britain

On the influence of Coriolis force on onset of thermal convection

By P. P. NIILER* AND F. E. BISSHOPP
Division of Applied Mathematics, Brown University, Providence, Rhode Island

(Received 1 September 1964 and in revised form 29 January 1965)

The effect of Coriolis force on the onset of thermal convection in a shallow layer of viscous liquid is studied in the limit of rapid rotation of the layer. Existing results which apply when the fluid lies between free boundaries are extended to the case where the boundaries are rigid.

1. Introduction

Benard (1900) observed that if a thin horizontal layer of liquid were heated from below then, at a critical temperature gradient across the layer, the initially stagnant layer would break up into cellular patterns of convective motion. The determination of the value of the temperature gradient at the onset of such motion is a problem which has been treated by various authors; an exhaustive treatment is given, for example, by Pellew & Southwell (1940). Their analysis is valid within the Boussinesq approximation to the Navier–Stokes equations, where the effect of density variations are neglected everywhere except when they appear in conjunction with the gravitational acceleration. Many further results are given by Chandrasekhar (1961) for this and related problems. The stability of the equilibrium configuration of the stagnant layer is determined by the Rayleigh number $R = g\alpha\beta d^4/\kappa\nu$, which can be thought of as the ratio of buoyant gravity forces (which tend to cause the instability) to frictional viscous forces (which tend to retard the instability). Here we have used the notation: g is the gravitational acceleration, α is the coefficient of volume expansion, β is the temperature gradient across the layer and d is the depth of the layer. The constants κ, ν are the coefficients of thermal conductivity and kinematic viscosity, respectively. For $R < R_c$, the system is said to be stable to disturbances of infinitesimal amplitude, and for $R > R_c$ the linear theory of instability predicts that convective motions may ensue.

Of particular interest to those interested in geophysical applications has been the closely related problem where the heated layer is under the influence of Coriolis force in addition to gravity. If the layer is rotating with respect to an inertial frame of reference, the critical Rayleigh number, R_c, at which convective motions are first allowed will be a function of the Taylor number, or magnitude of Coriolis forces. The Taylor number, $T \equiv 4\Omega^2 d^4/\nu^2$ (where Ω is the rate of

* Now at Department of Applied Physics, Harvard University, Cambridge, Massachusetts.

rotation), is essentially the square of the ratio of Coriolis force to viscous frictional force. Numerical calculations for $R_c = R_c(T)$, for $T < 10^8$, have been carried out by Chandrasekhar (1961) for various boundary conditions at the plane surfaces bounding the liquid layer. However, the nature of the instability at large T when the boundaries are rigid has not been fully dealt with in the literature. It is our purpose in this note to investigate the effect of rigid boundaries on the onset of thermal convection in the asymptotic limit of large Taylor number. We point out that calculations for R_c when $T < 10^8$ have been done by the Rayleigh–Ritz method, which is essentially limited to finite values of T, while our solution is an asymptotic expansion in T and is valid for $T > 10^{12}$.

The thermal instability problem for the rotating layer can be formulated in terms of linear, ordinary differential equations with constant coefficients. The exact solution of these equations can be written in terms of exponential functions, from which one obtains the asymptotic expansion of the solution. The two principal, and exact, results of this analysis are that in the limit of large Taylor number the viscous effects which are introduced at the boundary play a significant role only in an exponentially thin layer near the boundary, and hence the critical Rayleigh number is independent of whether the boundaries are rigid or free; and that in the major part of the fluid the gradient of the perturbations is constrained by rapid rotation to lie essentially in the direction normal to the axis of rotation.

2. Equations of the problem

For the present purpose, we shall refer to Chandrasekhar's formulation of the equations governing marginal stability of a rotating horizontal liquid layer which is heated from below. We shall assume that exchange of stabilities holds, though it has in fact been shown correct for sufficiently large Prandtl numbers only when the boundaries are free surfaces.

In terms of the dimensionless variables

$$w(z) = w'(z)/\Omega d, \quad \xi(z) = \nu\xi'(z)/2\Omega^2 d^2, \quad z = z'/d, \quad a = a'd, \tag{2.1}$$

which represent vertical components of velocity and vorticity, height and wavenumber of the perturbation of the steady state respectively, we can write the governing equations as

$$[D^2 - a^2]^3 w + TD^2 w = -Ra^2 w, \tag{2.2}$$

$$[D^2 - a^2]\,\xi + Dw = 0, \tag{2.3}$$

where D is written for d/dz. The temperature perturbation has not been neglected, simply eliminated at this stage of the formulation. The boundary conditions to be specified at $z = 0, 1$ for a free boundary are that the normal component of velocity, the viscous stress and the temperature perturbation vanish, that is,

$$w = 0, \quad D^2 w = 0, \quad D\xi = 0, \quad D^4 w = 0. \tag{2.4}$$

For a rigid boundary they are that the velocity and temperature perturbation should vanish, i.e.

$$w = 0, \quad Dw = 0, \quad \xi = 0; \quad (D^2 - a^2)^2 w - TD\xi = 0. \tag{2.5}$$

The eigenvalue problem defined above will yield a characteristic equation for the Rayleigh number $R = R(T, a)$, and we want to determine the minimum value of R_c for fixed T, when T is very large. Since the equations for $w(z)$, $\xi(z)$ are linear with constant coefficients, we can obtain the exact eigenfunction in terms of exponential functions of a complex argument, and, by substitution into the boundary conditions, the exact characteristic equation. For this reason we shall proceed to develop the asymptotic expansion of the characteristic equation first, and then apply the result of this expansion to the exact eigenfunctions to determine their asymptotic form. We point out that an alternate method of solution of the problem follows by construction of the asymptotic forms of the perturbation equations, the solutions of which should be the asymptotic eigenfunctions. To determine the asymptotic characteristic equation these solutions are substituted into the boundary conditions, which have to be satisfied to at least the same degree of approximation as the asymptotic differential equations. Because it is difficult to prove that such a scheme will indeed lead to the asymptotic expansion of the exact solution of the complete problem, we shall begin with what we know is the exact solution and compute its asymptotic expansion directly.

3. Representation of the solution

It is convenient to deal separately with the odd and even solutions of (2.2) and (2.3). Since the odd mode can be shown to be more stable (cf. Chandrasekhar 1961), we consider the even mode, which will be written in terms of the variable

$$\hat{z} = z - \tfrac{1}{2}. \tag{3.1}$$

The boundary conditions are now applied at $\hat{z} = \pm\frac{1}{2}$. The solution for $w(\hat{z})$, even in $\hat{z}$, is the real part of the complex representation

$$w(\hat{z}) = \sum_{i=1}^{3} A_i \cos\beta_i\hat{z}/\cos(\tfrac{1}{2}\beta_i). \tag{3.2}$$

The complex numbers β_i are the roots of the equation

$$(\beta_i^2 + a^2)^3 + \beta_i^2 T = a^2 R \quad (i = 1, 2, 3), \tag{3.3}$$

where the three roots of (3.3) are those which have the real part positive; A_i are complex-valued constants. The solution for $\xi(z)$ follows from (2.3)

$$\xi(\hat{z}) = A_0 \sinh a\hat{z}/\cosh(\tfrac{1}{2}a) - \sum_{i=1}^{3} A_i\beta_i \sin\beta_i\hat{z}/(a^2+\beta_i^2)\cos(\tfrac{1}{2}\beta_i). \tag{3.4}$$

The expressions (3.2) and (3.4) are to be substituted into (2.4) or (2.5), which, together with (3.3), will yield the exact form of the characteristic equation $R = R(a, T)$.

4. Free boundaries

In the case of free boundaries we take the result given by Chandrasekhar (1961); he finds that the critical Rayleigh number is proportional to $T^{\frac{2}{3}}$ and the corresponding wave-number of the most unstable disturbance is proportional to $T^{\frac{1}{6}}$. His solution is

$$R_c \sim P_c T^{\frac{2}{3}} \quad \text{and} \quad a_c \sim \alpha_c T^{\frac{1}{6}}, \tag{4.1}$$

where
$$P_c = (\alpha_c^6+\pi^2)/\alpha_c^2 = (27\pi^4/4)^{\frac{1}{3}} \tag{4.2}$$
and
$$\alpha_c = (\tfrac{1}{2}\pi^2)^{\frac{1}{6}}, \tag{4.3}$$
as $T \to \infty$. The eigenfunction can be obtained from equation (3.2); it follows easily that
$$w(z) \sim \sin \pi z. \tag{4.4}$$
The interesting feature of this case is that the asymptotic eigenfunction happens to be identical with the solution of the asymptotic differential equation,
$$(D^2-\alpha^6+\alpha^2P)\, w \sim 0, \tag{4.5}$$
for all values of z in [0, 1]. Consequently the solution of the asymptotic differential equation satisfies all the boundary conditions even though that equation is of lower order than the system defined by equations (2.2) and (2.3). In other words the free-boundary conditions give a solution in which, by accident, there is no boundary layer in the solution for w.

Now the asymptotic expansion of $w(z)$ indicates that the first-order term for the free boundaries is obtained by neglecting terms of $O(T^{-\frac{1}{3}})$ uniformly throughout the layer. In this case, the gradient of w makes an angle of $O(T^{-\frac{1}{6}})$ with the direction normal to the axis of rotation throughout the fluid region except in layers of $O(T^{-\frac{1}{3}})$ adjacent to the boundaries. In such regions, however, w is $O(T^{-\frac{1}{3}})$ and we have neglected such terms in our expansion. Hence, in regions where w is of significant magnitude, its gradient will be essentially in the direction normal to the axis of rotation. As will be shown in the next section, this will not be the case when the boundaries are rigid. In that case we will have to retain terms of $O(T^{-\frac{1}{12}})$ for the first-order solution while terms of $O(T^{-\frac{1}{6}})$ are neglected uniformly throughout the layer. In addition to a flow in the central part of the layer, there will be boundary layers of $O(T^{-\frac{1}{4}})$ adjacent to rigid walls, where w is of $O(T^{-\frac{1}{12}})$. In the major portion of the layer the gradient of w will again make an angle of $O(T^{-\frac{1}{6}})$ with the horizontal. In the boundary layer, however, it will make an angle of $O(T^{-\frac{1}{12}})$ with the direction of the axis of rotation, and hence will be more nearly in the direction of rotation.

At this point we shall not examine the cell structure in greater detail (cf. Veronis 1959); instead we shall proceed with the analysis for the case of rigid boundaries.

5. Rigid boundaries

To obtain the characteristic equation in the case of rigid boundaries, the solutions (3.2) and (3.4) are substituted into (2.5). We introduce the notation
$$\gamma_i = \beta_i \tan(\tfrac{1}{2}\beta_i) \quad (i = 1,2,3), \tag{5.1}$$
and the characteristic equation can then be written as
$$\operatorname{Im}\left[(\gamma_1-\gamma_2)\left(\frac{\gamma_1-\gamma}{a^2+\beta_1^2}-\frac{\overline{\gamma}_2-\gamma}{a^2+\overline{\beta}_2^2}\right)\right] = 0. \tag{5.2}$$
Here Im denotes the imaginary part, the bar denotes the complex conjugate, and
$$\gamma = (aR/T)\tanh(\tfrac{1}{2}a). \tag{5.3}$$
The details of the derivation of this expression and subsequent simplifications are found in appendix 1.

In the limit $T \to \infty$, we again expect that $a \to T^{\frac{1}{6}}$, and $R \to PT^{\frac{2}{3}}$. The roots of (3.3) can now be obtained by directly solving the cubic equation

$$\beta_1^2 = \alpha^2 P - \alpha^6 + O(T^{-\frac{1}{3}}), \quad \beta_2 = \sqrt{i} T^{\frac{1}{4}} [1 + O(T^{-\frac{1}{6}})], \tag{5.4}$$

The following asymptotic relations will also be useful:

$$\tanh(\tfrac{1}{2}a) = 1 + O\{\exp(-T^{\frac{1}{6}})\}, \quad \tanh(\tfrac{1}{2}\beta_2) = i + O\{\exp(-T^{\frac{1}{4}})\}. \tag{5.5}$$

The asymptotic expansion of expression (5.2) can now be obtained with the help of (5.3)–(5.5)

$$\beta_1 \tan(\tfrac{1}{2}\beta_1) = \sqrt{2}\,\alpha^2 T^{\frac{1}{12}} + O(1), \tag{5.6}$$

where we have retained the highest-order terms and used the definition (5.1). The smallest root of (5.6) is

$$\beta_1 = \pi(1 - \sqrt{2}/\alpha^2 T^{\frac{1}{12}}) + O(T^{-\frac{1}{6}}). \tag{5.7}$$

The expression for P, and hence the characteristic equation for R, i.e. $P \equiv R/T^{\frac{2}{3}}$, is obtained from (3.3), for $i = 1$, in the limit $T \to \infty$,

$$\alpha^2 P = \alpha^6 + \pi^2 - 2\pi^2 \sqrt{2}/\alpha^2 T^{\frac{1}{12}} + O(T^{-\frac{1}{6}}). \tag{5.8}$$

It is worth pointing out that the result (5.8) to $O(1)$ is identical to that of the free boundaries. However, to $O(T^{-\frac{1}{12}})$, (5.3) predicts a lower value for R_c than is obtained for the case of free boundaries.

The eigenfunction for the rigid boundaries can now be calculated from (3.2) and (3.4). In terms of the boundary-layer co-ordinate

$$\tilde{z} = \sqrt{2}\, T^{\frac{1}{4}}(\tfrac{1}{2} - |\hat{z}|), \tag{5.9}$$

$w(\hat{z})$ can be written as

$$w(\hat{z}) = A\{\cos\beta_1 \hat{z} - \pi e^{-\tilde{z}} \cos(\tilde{z} - \tfrac{1}{4}\pi)/\alpha^2 T^{\frac{1}{12}}\} + O(T^{-\frac{1}{6}}), \tag{5.10}$$

where A is an arbitrary constant.

For any $\hat{z}$ bounded away from the boundaries, the eigenfunction (5.10) is, to $O(1)$, identical with that of the free boundary solution of §4. However, for $\hat{z}$ approaching the boundary, i.e. $T \to \infty$ with $\tilde{z}$ fixed, there exists a boundary-layer flow which has a vertical component

$$\frac{w(\tilde{z})\,\alpha^2 T^{\frac{1}{12}}}{\pi} \sqrt{2} \sim A\{1 - e^{-\tilde{z}}(\cos\tilde{z} + \sin\tilde{z})\}. \tag{5.11}$$

Thus it is now apparent that in general we may expect two distinct limits of (5.6). The motion is dominated by constraints imposed by rapid rotation in the interior of the cell, and in addition there are boundary layers adjacent to the rigid walls where the asymptotic expansion of (2.2) is

$$\left(\frac{d^6}{d\tilde{z}^6} + 4\frac{d^2}{d\tilde{z}^2}\right) w(\tilde{z}) = O(T^{-\frac{1}{6}}). \tag{5.12}$$

Since the fluid particles move in tightly wound spirals in the interior of the fluid layer, and they must have zero velocity at the boundaries, (5.11) represents an Ekman boundary layer. We note that for $T \to \infty$ the motions which are induced by the thermal instability at the onset of steady convection are *small* motions departing from rigid body rotation, and that similar boundary-layer problems have been discussed by Proudman (1956), Stewartson (1957) and Robinson (1959).

Calculations for the characteristic equation $R = R(a, T)$ which yield a numerically converging critical Rayleigh number R_c for $T < 10^8$ have been carried out by Chandrasekhar (1961). From these calculations Chandrasekhar infers that as $T \to \infty$ the value of P_c should depend upon boundary conditions. According to the asymptotic expansion of the exact solution, however, P_c is independent of the boundary conditions to $O(1)$. For $T \sim 10^8$, one would expect a difference of

T	P_c, variational methods	P_c, asymptotic method
∞	—	8·69
10^{12}	—	7·80
10^{10}	7·51	7·27
10^8	7·09	6·63
10^6	7·11	5·62

TABLE 1. Comparison of values of P_c computed by variational methods with the values computed by asymptotic expansion of the exact solution for a rotating plane layer heated from below with rigid boundaries ($P_c \equiv R_c/T^{\frac{2}{3}}$).

the order of 10 % between P_c for rigid and free boundaries and this is sufficient to account for the difference observed in numerical calculations. To $O(T^{-\frac{1}{12}})$, only an exponentially thin layer close to the wall differentiates the solutions for a free boundary from those for a rigid boundary. Hence, from physical considerations alone, we would expect that, to $O(1)$, P_c would be independent of viscous boundary conditions, which is, of course, verified by the above limit analysis. The value of P_c to $O(1)$ is determined by the stipulation that the total mass of the system be conserved. Viscous effects introduced at the boundaries play a significant role only in determining the form of the asymptotic motion near the boundary. Equation (5.8) predicts the right trend for the critical Rayleigh number for the case of rigid boundaries for $T \gtrsim 10^{12}$. Table 1 is a numerical comparison of (5.8) with results quoted by Chandrasekhar, and for $T \sim 10^{10}$ the agreement is still remarkably good.

6. Concluding remarks

Perhaps the most significant result of this analysis is the observation that, in the limit of large Taylor number, the Rayleigh number (or the force needed to move the fluid parallel to rotation) is independent of the surfaces which bound the fluid. It was found that the effect of boundaries is constrained to extremely thin Ekman layers adjacent to these boundaries, and that the buoyancy force per unit volume required to produce small convective motions in the direction of rotation is $\rho g \alpha \beta \sim \Omega^{\frac{4}{3}}/\nu^{\frac{1}{3}}$. Hence the following phenomenon can be anticipated: were one to perform an experiment in a rapidly rotating layer, where a solid body is moved slowly in the direction of rotation, it can be conjectured that the drag coefficient will greatly increase with the rate of rotation. This is due to the fact that at rapid rotation, or small viscosity, the vortex lines become more strongly bound to the fluid and stretching them with small motions becomes increasing difficult.

Since rapid rotation causes a marked simplification of the motion in a large part of the fluid layer (cf. equation (4.8)), it is hoped that various related problems will receive some impetus from this analysis. For example, a simple extension of the above method would also yield the relevant answers very quickly for the horizontal rapidly rotating layer with quite an arbitrary initial temperature gradient. In a subsequent paper, this analysis will be used to motivate the asymptotic approach for dealing with the thermal instability problem in a rapidly rotating, self-gravitating sphere.

From physical considerations, two basic limitations are imposed on this analysis. The assumption of the validity of the principle of exchange of stabilities may not be made for an arbitrary value of Prandtl number; the instability occurs as over-stability rather than steady convection when the Prandtl number is small. Because of the great similarity of the free and rigid boundary eigenfunctions throughout most of the layer, it follows, though we shall not show it here, that the Prandtl number above which convection appears for rigid boundaries differs from the result obtained by Chandrasekhar for free boundaries by $O(T^{-\frac{1}{12}})$. Secondly, in our analysis we have considered the density variations in conjunction only with gravitational accelerations. This is the essence of the Boussinesq approximation to the Navier–Stokes equations in a non-rotating, shallow layer of fluid. If the heated fluid rotates, centrifugal accelerations will also be coupled with the density variations, and we have neglected this effect. Hence, we must consider our analysis strictly valid when the Prandtl number is large (cf. Chandrasekhar 1961) and $\Omega^2 X_0/g$ is small, where X_0 is the horizontal extent of the layer.

The results presented in this note were obtained in the course of research sponsored in part by the Office of Naval Research under contract with Brown University.

REFERENCES

BENARD, H. 1900 *Review Gén. Sci. Pur. Appl.* **12**, 1261.

CHANDRASEKHAR, S. 1961 *Hydrodynamic and Hydromagnetic Stability*. Oxford University Press.

PELLEW, A. & SOUTHWELL, R. V. 1940 *Proc. Roy. Soc.* A, **176**, 312.

PROUDMAN, I. 1956 *J. Fluid Mech.* **1**, 505.

ROBINSON, A. R. 1959 *J. Fluid Mech.* **6**, 599.

STEWARTSON, K. 1957 *J. Fluid Mech.* **3**, 17.

VERONIS, G. 1959 *J. Fluid Mech.* **5**, 401.

Appendix 1

The expressions for the rigid boundary conditions at $\hat{z} = \pm \frac{1}{2}$ are obtained by substituting equations (3.2) and (3.4) into (2.5). These conditions are

$$\left.\begin{aligned}
&\sum_{i=1}^{3} A_i = 0, \\
&\sum_{i=1}^{3} A_i \beta_i \tan(\tfrac{1}{2}\beta_i) = 0, \\
&\sum_{i=1}^{3} A_i \beta_i \tan(\tfrac{1}{2}\beta_i)/(a^2+\beta_i^2) - A_0 \tanh(\tfrac{1}{2}a) = 0, \\
&\sum_{i=1}^{3} A_i/(a^2+\beta_i^2) - A_0 T/Ra = 0,
\end{aligned}\right\} \tag{A 1}$$

where we have used the result (3.3) in the last expression of (A 1).

The characteristic equation is formed by setting the determinant of the coefficients A_i $(i = 0, 1, 2, 3)$ equal to zero

$$\begin{vmatrix}
1 & 1 & 1 & 0 \\
\beta_1 \tan(\frac{1}{2}\beta_1) & \beta_2 \tan(\frac{1}{2}\beta_2) & \beta_3 \tan(\frac{1}{2}\beta_3) & 0 \\
\dfrac{\beta_1 \tan(\frac{1}{2}\beta_1)}{a^2+\beta_1^2} & \dfrac{\beta_2 \tan(\frac{1}{2}\beta_2)}{a^2+\beta_2^2} & \dfrac{\beta_3 \tan(\frac{1}{2}\beta_3)}{a^2+\beta_3^2} & -\tanh(\frac{1}{2}a) \\
\dfrac{1}{a^2+\beta_1^2} & \dfrac{1}{a^2+\beta_2^2} & \dfrac{1}{a^2+\beta_3^2} & \dfrac{T}{Ra}
\end{vmatrix} = 0. \tag{A 2}$$

We now expand (A 2) in the minors of the fourth column

$$\begin{vmatrix}
1 & 1 & 1 \\
\gamma_1 & \gamma_2 & \gamma_3 \\
\dfrac{\gamma_1}{a^2+\beta_1^2} & \dfrac{\gamma_2}{a^2+\beta_2^2} & \dfrac{\gamma_3}{a^2+\beta_3^2}
\end{vmatrix} = \gamma \begin{vmatrix}
1 & 1 & 1 \\
\gamma_1 & \gamma_2 & \gamma_3 \\
\dfrac{1}{a^2+\beta_1^2} & \dfrac{1}{a^2+\beta_2^2} & \dfrac{1}{a^2+\beta_3^2}
\end{vmatrix}, \tag{A 3}$$

where
$$\gamma = aR \tanh(\tfrac{1}{2}a)/T \tag{A 4}$$
and
$$\gamma_i = \beta_i \tan(\tfrac{1}{2}\beta_i) \quad (i = 1, 2, 3). \tag{A 5}$$

It follows from (3.3) that β_1 and β_2 are complex conjugates, which implies that γ_1 and γ_2 are complex conjugates also. With this observation, we find that (A 3) can be further reduced by subtracting the first column from the second and third columns and expanding both determinants in minors of the first row. This calculation yields

$$\mathrm{Im}\left\{(\gamma_1-\gamma_2)\left(\frac{\gamma_1-\gamma}{a^2+\beta_1^2}-\frac{\gamma_3-\gamma}{a^2+\beta_3^2}\right)\right\} = 0. \tag{A 6}$$

In the limit $T \to \infty$ we use the asymptotic relations

$$R \to PT^{\frac{2}{3}}, \quad a \to \alpha T^{\frac{1}{6}} \tag{A 7}$$

in calculating the roots of (3.3)

$$\beta_1^2 = P\alpha^2 - \alpha^6 + O(T^{-\frac{1}{6}}), \quad \beta_2 = \sqrt{i}T^{\frac{1}{4}}[1+O(T^{-\frac{1}{6}})], \quad \beta_3 = \bar{\beta}_2. \tag{A 8}$$

The asymptotic expansions of γ, γ_1, γ_2 now follow as

$$\gamma \rightarrow \alpha P/T^{\frac{1}{6}}, \quad \gamma_2 \rightarrow T^{\frac{1}{4}} i \sqrt{i}, \quad \gamma_3 = \overline{\gamma}_2, \tag{A 9}$$

where we have used

$$\tanh\left(\tfrac{1}{2}a\right) = 1 + O(\exp\left[-T^{\frac{1}{6}}\right]), \quad \tanh\left(\tfrac{1}{2}\beta_2\right) = i + O(\exp\left[-T^{\frac{1}{4}}\right]). \tag{A 10}$$

In (A 8) and (A 9) the square root is chosen which has its real part positive. We use the results (A 7)–(A 10) in (A 6) and retain the highest-order terms in the limit $T \rightarrow \infty$ to obtain the characteristic equation in the form

$$\gamma_1 = \sqrt{2}\,\alpha^2 T^{\frac{1}{12}}. \tag{A 11}$$

To obtain the lowest root of (A 11) we substitute the expansion

$$\beta_1 = \beta_1^{(0)} + \beta_1^{(1)}/T^{\frac{1}{12}} + O(T^{-\frac{1}{6}}) \tag{A 12}$$

into (A 5) and (A 12). This gives

$$\beta_1^{(0)} = \pi, \quad \beta_1^{(1)} = -\pi\sqrt{2}/\alpha^2. \tag{A 13}$$

The characteristic equation for the proportionality constant, $P = R/T^{\frac{2}{3}}$, is obtained from (A 7), (A 12) and (A 13) as

$$\alpha^2 = \pi^2 + \alpha^6 - 2\sqrt{2}\pi^2\alpha^2/(T^{\frac{1}{12}}) + O(T^{-\frac{1}{6}}). \tag{A 14}$$

The minimum value of P occurs where $\partial P/\partial \alpha = 0$. The value α_c for P_c is obtained by substituting the expansion

$$\alpha_c = \alpha_c^{(0)} + \alpha_c^{(1)}/T^{\frac{1}{12}} + O(T^{-\frac{1}{6}}) \tag{A 15}$$

into the derivative of (A 14) and equating like powers of T in the resulting expression. It then follows that

$$\alpha_c^{(0)} = \left(\tfrac{1}{2}\pi^2\right)^{\frac{1}{6}}, \quad a_c^{(1)}\alpha_c^{(0)} = -\tfrac{2}{3}\sqrt{2}, \tag{A 16}$$

and the critical value of Rayleigh number, in the limit $T \rightarrow \infty$, is calculated from

$$P_c = (\alpha_c^{(0)})^4 + \pi^2/(\alpha_c^{(0)})^2 - 4\sqrt{2}(\alpha_c^{(0)})^2/T^{\frac{1}{12}}. \tag{A 17}$$

The results of this calculation for $T > 10^6$ are summarized in table 1.